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A note on measuring inequality and welfare for ordinal data

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Abstract

Many countries (e.g. UK, France, Japan, Canada) and international institutions have recently adhered to measuring progress beyond GDP per capita. This, however, often requires dealing with ordinal data. Therefore inequality measurement theory for ordinal data is being developed in the last decade, because standard inequality measures cannot be used for ordinal data; the same applies to measuring welfare (Kobus and Milos, 2012). So far researchers have been focused on developing the dominance criterion proposed by Allison and Foster (2004) (henceforth AF). Yet this criterion seems to be better suited to measuring polarization, and not necessarily inequality, as the two notions are different in a cardinal data setting (Esteban and Ray 1994). Due to limitations of AF approach, welfare measurement, which is related to inequality measurement, has not been developed yet for ordinal data. This note is our preliminary work on measuring inequality and welfare for ordinal data. We propose a framework which can encompass both notions, and within which inequality is different from polarization. In this note we shed light on how standard concepts and theorems can be redefined in our framework.

Keywords: Inequality measurement; Ordinal data; Majorization; Sub-/super-modular function; Zeta function; Mobius inversion matrix;

JEL codes: D3; D6

1 Notation and definitions

$x^T = (p_1, p_2, ..., p_n)$ such that $p_1 + p_2 + ... + p_n = 1$

Definition 1. ***Classical Majorization***

For $x, y \in \mathbb{R}^n_+$,

$$x \prec y \qquad \textit{iff} \qquad \begin{cases} \Sigma_{i=1}^k x_{[i]} \leqslant \Sigma_{i=1}^k y_{[i]} & k = 1, 2, ..., (n-1) \\ \Sigma_{i=1}^n x_i = \Sigma_{i=1}^n y_i \end{cases}$$

Where $(z_{[1]}, z_{[2]}, ..., z_{[n]}) = sort \downarrow (\boldsymbol{z})$ *denoted descendingly sorted (nonincreasing) ordering of* $\boldsymbol{z} = (z_1, z_2, ..., z_n)$

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Definition 2. ***Generalized Majorization***

For $x, y \in \mathbb{R}^n_+$,

$$x \preceq y \qquad \text{iff} \qquad \begin{cases} \Sigma_{i=1}^k x_i \leqslant \Sigma_{i=1}^k y_i & k = 1, 2, ..., (n-1) \\ \Sigma_{i=1}^n x_i = \Sigma_{i=1}^n y_i \end{cases}$$

Definition 3. ***Vector Majorization***

For $x, y \in \mathbb{R}^n_+$,

$$x \leqslant_{vec} y \qquad \text{iff} \qquad x_i \leqslant y_i \qquad k = 1, 2, ..., (n-1)$$

Definition 4. *Partial sum matrix* ($\int$) *is a lower triangular matrix consisting of only* 1, *i.e.* $\int = (\zeta_{ij})$ *where* $\zeta_{ij} = 0$ *if* $j > i$ *and* $\zeta_{ij} = 1$ *elsewhere. For example, when* $n = 6$*:*

$$\int = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Definition 5. ***Generalized Majorization (equivalent definition)***

For $x, y \in \mathbb{R}^n_+$,

$$x \preceq y \qquad \text{iff} \qquad \int x \leqslant_{vec} \int y \qquad \text{and} \qquad \Sigma_{i=1}^n x_i = \Sigma_{i=1}^n y_i$$

Definition 6. *Mobius inversion matrix* (∂) *is a lower triangular matrix consisting of* 1 *on diagonal,* -1 *on neighbourhood elements below diagonal and 0 elsewhere, i.e.* $\partial = (\mu_{ij})$ *where* $\mu_{ij} = 1$ *if* $j = i$, $\mu_{ij} = -1$ *if* $j + 1 = i$ *and* $\mu_{ij} = 0$ *elsewhere. For example, when* $n = 6$*:*

$$\int = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

We denote that $\partial = \int^{-1}$

Definition 7. ***Lower Triangular Stochastic Matrix (LTSM)***

Matrix is column stochastic if its elements are all nonnegative and all columns add up to 1.

LTSM is column stochastic matrix which has only zero elements above diagonal.

Example:

$$\begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & 0 & 0 \\ \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix}$$

Definition 8. ***LTSM Majorization***

For $x, y \in \mathbb{R}^n_+$,

$x \preceq_{LTSM} y$ *iff* $x = Ly$ *for some LTSM* L

2 LTSM and generalized majorization

Theorem 1. *LTSM Majorization* $\iff$ *Generalized majorization*

Proof:

$\Rightarrow$

Let $x \preceq_{LTSM} y$, then $x = Ly$ for some LTSM L, then $\int x = \int Ly$, which implies $\int x \leqslant_{vec} \int y$[1] and we have $x \preceq y$.

$\Leftarrow$

For the converse we assume $x \preceq y$ and derive an LTSM L such that $x = Ly$. Let's assume $x \neq y$, otherwise L would be identity matrix. Let's consider:

Definition 9. ***Exchange matrix*** $L_{pq}(\varepsilon)$ *which differs from identity only on* $\{p, q\} \times \{p, q\}$

$$\begin{pmatrix} \dots & p & \dots & q & \dots \end{pmatrix} \begin{pmatrix} \dots \\ p \\ \dots \\ q \\ \dots \end{pmatrix} \begin{pmatrix} \ddots & & & & \\ & 1-\varepsilon & & & \\ & & \ddots & & \\ & \varepsilon & & 1 & \\ & & & & \ddots \end{pmatrix}$$

Now let's find lowest p, q such that $x_p \neq y_p$ and $x_q \neq y_q$ (so we have $x_1 = y_1, ..., x_{p-1} = y_{p-1}$). There has to exist at least 2 such elements because if their sum is equal, they can't differ on only 1 position. Recall that we have $x \preceq y$ so $x_p \leqslant y_p$ and $\frac{x_p}{y_p} \in (0, 1]$. We define $\varepsilon = (1 - \frac{x_p}{y_p}) \in [0, 1)$. We have $y^* = L_{pq}(\varepsilon)y = (y_1, y_2, ..., y_{p-1}, x_p, y_{p+1}, ..., y_{q-1}, (y_q - y_p + x_p), y_{q+1}, ..., y_n)$. Since $x_p \leqslant y_p$ we have also $x \leqslant_{vec} y^* \leqslant_{vec} y$ (so multiplication by $L_{pq}(\varepsilon)$ preserves majorization) and $y*$ agrees with

[1] $x = Ly = (l_{11}y_1, l_{21}y_1 + l_{22}y_2, ..., l_{n1}y_1 + l_{n2}y_2 + ... + l_{n1}y_n)$, $\int x = (l_{11}y_1, [l_{11} + l_{21}]y_1 + l_{22}y_2, ..., [l_{11} + l_{21} + .. + l_{n1}]y_1 + [l_{22} + l_{32} + ... + l_{n2}]y_2 + ... + l_{n1}y_n)$ and we clearly have $\int x \leqslant_{vec} \int y$ because each of $[l_{1m} + l_{2m} + .. + l_{km}] \leqslant 1$, because L is columnt stochastic. Additionally we obtain that $\Sigma_{i=1}^n x_i = [l_{11} + l_{21} + .. + l_{n1}]y_1 + [l_{22} + l_{32} + ... + l_{n2}]y_2 + ... + l_{n1}y_n = \Sigma_{i=1}^n y_i$

x on one more beggining position than y. We can now apply our algoritm as an induction step and by at most n exchanges we will obtain x ($x = y^{*n}$).

Lemma 1. *LTSM form a semigroup.*

Proof:

It is clear that identity matrix belongs to LTSM. Now we only need to check that product of 2 LTSM matrices is again LTSM matrix. Let L and K be LTSM matrices.
We have $(LK)_{ij} = l_{i1}k_{1j} + l_{i2}k_{2j} + ... + l_{in}k_{nj}$. Now remember that $l_{im} = 0$ for $m > i$ and $k_{mj} = 0$ for $m < j$, so if we want to obtain nonzero entries we need to have $j \leqslant m \leqslant i$. So $(LK)_{ij} = 0$ for $j > i$ (LK is lower triangular) and $(LK)_{ij} = l_{ij}k_{jj} + l_{i(j+1)}k_{(j+1)j} + l_{i(i-1)}k_{(i-1)j} + l_{ii}k_{ij}$ for $j \leqslant i$.
Let's check if LK is column stochastic:
$\Sigma_{k=1}^{n}(LK)_{kj} = \Sigma_{x=j}^{n}(LK)_{xj} = \Sigma_{x=j}^{n}\Sigma_{m=j}^{x}l_{xm}k_{mj} =^{2} \Sigma_{m=j}^{n}k_{mj}(\Sigma_{x=m}^{n}l_{xm}) = \Sigma_{m=j}^{n}k_{mj} = 1$ since both $\Sigma_{m=j}^{n}k_{mj}$ and $\Sigma_{x=m}^{n}l_{xm}$ are equal 1 due to L and K being an LTSM.
We can now say that $x = Ly = L_{(n-1)n}(\varepsilon_{n-1})...L_{23}(\varepsilon_2)L_{12}(\varepsilon_1)y$ by which we finish our proof.

Corollary 1. *Each LTSM can be decomposed into product of at most $n-1$ exchange matrices and this semigroup is generated by exchange matrices.*

3 Generalized Majorization of nth order

Definition 10. ***Generalized Majorization of nth order***
For $x, y \in \mathbb{R}_+^n$,
$x \preceq^n y$ *iff* $\int^n x \leqslant_{vec} \int^n y$ *and* $(\int^n x)_n = (\int^n y)_n$

Now let's take $x^* = \int^{n-1} x$ and $y^* = \int^{n-1} y$. We obtain $\int x^* \leqslant_{vec} \int y^*$. By Theorem 1 we know that since $x^* \preceq y^*$ there exist LTSM L such that $x^* = Ly^*$. So $x = \partial^{n-1} L \int^{n-1} y$.

Definition 11. ***Semigroups of nth order related to LTSM***
Let's define $G^k = \{\partial^{k-1} L \int^{k-1} | L \in LTSM\}$

It is clear that G^k forms a semigroup because $\partial = \int^{-1}$.

Theorem 2. *Each $L^k \in G^k$ can be decomposed into at most $n-1$ matrices related to exchange matrices.*

Proof:

[2]we regroup terms by k

Let's take

$$G^k \ni L^k = \partial^{k-1} L \int^{k-1} =$$

$$= \partial^{k-1} L_{(n-1)n}(\varepsilon_{n-1})...L_{23}(\varepsilon_2) L_{12}(\varepsilon_1) \int^{k-1} =$$

$$= \partial^{k-1} L_{(n-1)n}(\varepsilon_{n-1}) \int^{k-1} \partial^{k-1} ... \int^{k-1} \partial^{k-1} L_{23}(\varepsilon_2) \int^{k-1} \partial^{k-1} L_{12}(\varepsilon_1) \int^{k-1} =$$

$$= L^k_{(n-1)n}(\varepsilon_{n-1})...L^k_{23}(\varepsilon_2) L^k_{12}(\varepsilon_1)$$

4 Properties of exchange matrices under $\partial - \int$ transformation

Definition 12. $c_1(x) = 1$

$c_2(x) = c_1(1) + c_1(2) + ... c_1(x) = x$

$c_3(x) = c_2(1) + c_2(2) + ... c_2(x) = 1 + 2 + ... + x = \frac{1}{2} x(x+1)$

$\vdots$

$c_k(x) = c_{k-1}(1) + c_{k-1}(2) + ... + c_{k-1}(x) = \frac{1}{(k-1)!} x(x+1)(x+2)...(x+k-2)$[3]

$$\int^k = \begin{pmatrix} c_k(1) & 0 & 0 & 0 & 0 & \dots & 0 \\ c_k(2) & c_k(1) & 0 & 0 & 0 & \dots & 0 \\ c_k(3) & c_k(2) & c_k(1) & 0 & 0 & \dots & 0 \\ c_k(4) & c_k(3) & c_k(2) & c_k(1) & 0 & \dots & 0 \\ c_k(5) & c_k(4) & c_k(3) & c_k(2) & c_k(1) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_k(n) & c_k(n-1) & c_k(n-2) & c_k(n-3) & c_k(n-4) & \dots & c_k(1) \end{pmatrix}$$

Definition 13. ***Pascal's triangle coefficients***

$t_0(x)$ 1

$t_1(x)$ 1 1

$t_2(x)$ 1 2 1

$t_3(x)$ 1 3 3 1

$\vdots$

$t_k(x)$ $\binom{k}{0}$ $\binom{k}{1}$ $\binom{k}{2}$ $\binom{k}{3}$ $\dots$ $\binom{k}{k}$ *So we have* $t_k(x) = \binom{k}{x-1}$ *for* $1 \leqslant x \leqslant k+1$ *and* $t_k(x) = 0$ *for* $k+1 \leqslant x$

[3] Generating function for $c_1(n)$ is $\frac{1}{1-x}$, so giving the form of $c_k(n)$, its generating function will be $(\frac{1}{(1-x)})^k$, so $c_k(1+n) = \frac{1}{n!}((\frac{1}{(1-x)})^k)^{(n)}(0) = \frac{1}{n!}((1-x)^{-k})^{(n)}(0) = \frac{1}{n!} k(k+1)(k+2)...(k+n-1)((1-x)^{-k-n})(0) = \frac{1}{n!} k(k+1)(k+2)...(k+n-1) = \frac{(k+n-1)!}{n!(k-1)!} = \frac{1}{(k-1)!}(n+1)(n+2)...(n+k-1)$, so it agrees with our formula.

$$
\partial^k = \begin{pmatrix}
t_k(1) & 0 & 0 & 0 & 0 & \dots & 0 \\
-t_k(2) & t_k(1) & 0 & 0 & 0 & \dots & 0 \\
t_k(3) & -t_k(2) & t_k(1) & 0 & 0 & \dots & 0 \\
-t_k(4) & t_k(3) & -t_k(2) & t_k(1) & 0 & \dots & 0 \\
t_k(5) & -t_k(4) & t_k(3) & -t_k(2) & t_k(1) & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^{n+1}t_k(n) & (-1)^n t_k(n-1) & (-1)^{n-1}t_k(n-2) & (-1)^{n-2}t_k(n-3) & (-1)^{n-3}t_k(n-4) & \dots & t_k(1)
\end{pmatrix}
$$

Definition 14. ***General form of $L_{pq}(\varepsilon)$ matrix under $\partial - \int$ transformation***

$$
L_{pq}^{k+1}(\varepsilon) = \partial^k L_{pq}(\varepsilon) \int^k = Id +
$$

$$
\begin{array}{c} \vdots \\ p-1 \\ p \\ p+1 \\ p+2 \\ p+3 \\ \vdots \\ n \end{array}
\begin{pmatrix} \vdots \\ 0 \\ -t_k(1) \\ t_k(2) \\ -t_k(3) \\ t_k(4) \\ \vdots \\ (-1)^{n-p+1}t_k(n-p+1) \end{pmatrix}
\begin{pmatrix} c_k(p) & c_k(p-1) & \dots & c_k(3) & c_k(2) & c_k(1) & 0 & \dots & 0 \end{pmatrix} \varepsilon +
$$

$$
\begin{array}{c} \vdots \\ q-1 \\ q \\ q+1 \\ q+2 \\ q+3 \\ \vdots \\ n \end{array}
\begin{pmatrix} \vdots \\ 0 \\ t_k(1) \\ -t_k(2) \\ t_k(3) \\ -t_k(4) \\ \vdots \\ (-1)^{n-p}t_k(n-q+1) \end{pmatrix}
\begin{pmatrix} c_k(p) & c_k(p-1) & \dots & c_k(3) & c_k(2) & c_k(1) & 0 & \dots & 0 \end{pmatrix} \varepsilon
$$

Proof:

Let $D_{pq}(\varepsilon) = L_{pq}(\varepsilon) - Id$, i. e. $(D_{pq})_{pp} = -\varepsilon$, $(D_{pq})_{qp} = \varepsilon$ and $(D_{pq})_{ij} = 0$ elsewhere.
$L_{pq}^{k+1}(\varepsilon) = \partial^k L_{pq}(\varepsilon) \int^k = \partial^k (Id + D_{pq}(\varepsilon)) \int^k = \partial^k Id \int^k + \partial^k D_{pq}(\varepsilon) \int^k = Id + \partial^k D_{pq}(\varepsilon) \int^k$ and it immediatelly follows that L_{pq}^{k+1} is of the above form.

5 Examples

Example 1

$p = 3$, $q = 6$, n=9

$$L_{36}(\varepsilon)=\begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1-\varepsilon&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&\varepsilon&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{pmatrix}$$

$$L_{36}^2(\varepsilon)=\begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\-\varepsilon&-\varepsilon&1-\varepsilon&0&0&0&0&0&0\\\varepsilon&\varepsilon&\varepsilon&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\\varepsilon&\varepsilon&\varepsilon&0&0&1&0&0&0\\-\varepsilon&-\varepsilon&-\varepsilon&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{pmatrix}=Id+\begin{pmatrix}0\\0\\-1\\1\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&1&1&0&0&0&0&0&0\end{pmatrix}\varepsilon+$$

$$\begin{pmatrix}0\\0\\0\\0\\0\\1\\-1\\0\\0\end{pmatrix}\begin{pmatrix}1&1&1&0&0&0&0&0&0\end{pmatrix}\varepsilon$$

$$L_{36}^6(\varepsilon)=\begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\-15\varepsilon&-5\varepsilon&1-\varepsilon&0&0&0&0&0&0\\75\varepsilon&25\varepsilon&5\varepsilon&1&0&0&0&0&0\\-150\varepsilon&-50\varepsilon&-10\varepsilon&0&1&0&0&0&0\\165\varepsilon&55\varepsilon&11\varepsilon&0&0&1&0&0&0\\-150\varepsilon&-50\varepsilon&-10\varepsilon&0&0&0&1&0&0\\165\varepsilon&55\varepsilon&11\varepsilon&0&0&0&0&1&0\\-150\varepsilon&-50\varepsilon&-10\varepsilon&0&0&0&0&0&1\end{pmatrix}=Id+\begin{pmatrix}0\\0\\-1\\5\\-10\\10\\-5\\1\\0\end{pmatrix}\begin{pmatrix}15&5&1&0&0&0&0&0&0\end{pmatrix}\varepsilon+$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -5 \\ 10 \\ -10 \end{pmatrix} \begin{pmatrix} 15 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon$$

Example 2

$p = 5$, $q = 7$, n=9

$$L_{57}(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L_{57}^5(\varepsilon) = Id + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 4 \\ -6 \\ 4 \\ 1 \end{pmatrix} \begin{pmatrix} 35 & 20 & 10 & 4 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -4 \\ 6 \end{pmatrix} \begin{pmatrix} 35 & 20 & 10 & 4 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon =$$

$$Id+\begin{pmatrix}0\\0\\0\\0\\-1\\4\\-5\\0\\-5\end{pmatrix}\begin{pmatrix}35 & 20 & 10 & 4 & 1 & 0 & 0 & 0 & 0\end{pmatrix}\varepsilon = Id+\begin{pmatrix}0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\-35&-20&-10&-4&-1&0&0&0&0\\140&80&40&16&4&0&0&0&0\\-175&-100&-50&-20&-5&0&0&0&0\\0&0&0&0&0&0&0&0&0\\-175&-100&-50&-20&-5&0&0&0&0\end{pmatrix}\varepsilon =$$

$$\begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\-35\varepsilon&-20\varepsilon&-10\varepsilon&-4\varepsilon&1-\varepsilon&0&0&0&0\\140\varepsilon&80\varepsilon&40\varepsilon&16\varepsilon&4\varepsilon&1&0&0&0\\-175\varepsilon&-100\varepsilon&-50\varepsilon&-20\varepsilon&-5\varepsilon&0&1&0&0\\0&0&0&0&0&0&0&1&0\\-175\varepsilon&-100\varepsilon&-50\varepsilon&-20\varepsilon&-5\varepsilon&0&0&0&1\end{pmatrix}$$

Example 3

$p = 5$, $q = 6$, n=9

$$L_{56}(\varepsilon) = \begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1-\varepsilon&0&0&0&0\\0&0&0&0&\varepsilon&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{pmatrix}$$

$$L^2_{56}(\varepsilon) = \begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\-\varepsilon&-\varepsilon&-\varepsilon&-\varepsilon&1-\varepsilon&0&0&0&0\\2\varepsilon&2\varepsilon&2\varepsilon&2\varepsilon&2\varepsilon&1&0&0&0\\-\varepsilon&-\varepsilon&-\varepsilon&-\varepsilon&-\varepsilon&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{pmatrix}$$

$$L^3_{56}(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -5\varepsilon & -4\varepsilon & -3\varepsilon & -2\varepsilon & 1-\varepsilon & 0 & 0 & 0 & 0 \\ 15\varepsilon & 12\varepsilon & 9\varepsilon & 6\varepsilon & 3\varepsilon & 1 & 0 & 0 & 0 \\ -15\varepsilon & -12\varepsilon & -9\varepsilon & -6\varepsilon & -3\varepsilon & 0 & 1 & 0 & 0 \\ 5\varepsilon & 4\varepsilon & 3\varepsilon & 2\varepsilon & \varepsilon & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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